arXiv:solv-int/9501001v1 30 Dec 1994

The integrable dynamics of discrete and continuous curves*

Adam Doliwa
Institute of Theoretical Physics, Warsaw University
ul. Hoża 69, 00-681 Warsaw, Poland
e-mail: doliwa@fuw.edu.pl

Paolo Maria Santini
Dipartimento di Fisica, Università di Catania
Corso Italia 57, I-95129 Catania, Italy
and INFN, Sezione di Roma, P.le Aldo Moro 2, I-00185 Roma, Italy
e-mail: santini@catania.infn.it santini@roma1.infn.it

Abstract

We show that the following geometric properties of the motion of discrete and continuous curves select integrable dynamics: i) the motion of the curve takes place in the N dimensional sphere of radius R, ii) the curve does not stretch during the motion, iii) the equations of the dynamics do not depend explicitly on the radius of the sphere. Well known examples of integrable evolution equations, like the nonlinear Schrödinger and the sine-Gordon equations, as well as their discrete analogues, are derived in this general framework.

1 A historical introduction

One of the classical problems of the XIX-century geometers was the study of the connection between differential geometry of submanifolds and nonlinear (integrable) PDE's. For instance, Liouville found the general solution of the equation (known now as the Liouville equation) which describes minimal surfaces in E^3 [1]. Bianchi solved the general Goursat problem for the sine-Gordon (SG) equation [2], which encodes the whole geometry of the pseudospherical surfaces. Moreover the method of construction of a new pseudospherical surface from a given one, proposed by Bianchi [3], gives rise to the Bäcklund transformation for the SG equation [4].

The connection between geometry and integrable PDE's became even deeper when Hasimoto [5] found the transformation between the equations governing the curvature and torsion of a nonstretching thin vortex filament moving in an incompressible fluid and the NLS equation. Several authors, including Lamb [6], Lakshmanan [7], Sasaki [8], Chern and Tenenblat [9] related the Zakharov-Shabat(ZS) [10] spectral problem and the associated Ablowitz-Kaup-Newell-Segur(AKNS) hierarchy [11] to the motion of curves in E^3 or to the pseudospherical surfaces and certain foliations on them.

Almost at that time Sym introduced the soliton surfaces approach, in which the powerful tools of the IST method are used to construct explicit formulas for the immersions

*This work was supported by KBN grant 2-0168-91-01, by the INFN and by the 1994 agreement between Warsaw and Rome Universities

of one-parameter families (labeled by the spectral parameter) of surfaces corresponding to given solutions of integrable PDE's [12]; see also the recent developments of Bobenko [13].

More recently Langer and Perline [14] showed that the dynamics of a nonstretching vortex filament in R^3 gives rise, through the Hasimoto transformation, to the recursion operator of the NLS hierarchy. Similarly, Goldstein and Petrich [15] showed that the dynamics of a nonstretching string on the plane produces the recursion operator of the mKdV hierarchy.

Connections between geometry and integrable PDE's in multidimensions can also be found, for example in works of Tenenblat and Terng [16] and Konopelchenko [17]. Also at a discrete level there is a similar situation. For instance, discrete pseudospherical surfaces and the discrete analogues of constant mean curvature surfaces are described by integrable discrete analogues of the sine-Gordon (SG) [18][19] and sinh-Gordon equations [20]. Such discretizations were found by adapting the main geometric properties of the continuous surfaces to a discrete level.

In two recent papers [21][22] we have proposed a new geometric characterization of the integrable dynamics of a discrete or continuous curve (where, by discrete curve, we mean just a sequence of points), based on the following three properties:

Property 1. The motion of the curve takes place in the N-dimensional sphere of radius R, denoted by $S^N(R)$, $N > 1$.

Property 2. The curve does not stretch during the motion.

Property 3. The equations of the dynamics of the curve do not depend **explicitly** on the radius R.

We remark that *Properties 1-3* not only select integrable PDE's, but also provide their integrability scheme; in other words, in the process of deriving the dynamics selected by *Properties 1-3* one discovers "for free" the integrable nature of such dynamics! In particular, the spectral problem is given by the Frenet equations of the curve and is a consequence of *Property 1*, and the spectral parameter is given by the inverse of the radius of the sphere.

We also remark that our aproach explains in a simple way Sym's formula [12], which allows to calculate, from the wave function of the spectral problem, the surface generated by the motion of the curve.

In papers [21] and [22] we have dealt with the integrable dynamics of a continuous and discrete curve respectively, obtaining, in the case of $N = 3$, the AKNS [11] and the Ablowitz-Ladik (AL) [23] hierarchies. As we shall see in the following if, in *Property 3*, we consider discrete time dynamics, one also generates integrable fully discrete evolution equations, like the Hirota equation [24][25]. Since integrable discrete dynamics can always be interpreted as Bäcklund Transformations (BT's) of the corresponding continuous dynamics [26], the hierarchies of BT's of integrable systems are also characterized by *Properties 1–3.*

2 The curve in $S^N(R)$ and the associated spectral problem

2.1 Frenet basis along the discrete curve

Let us consider a sequence $\mathbf{Z} \ni k \mapsto \mathbf{r}(k) \in S^N(R) \subset \mathbf{R}^{N+1}$ of points of the N-dimensional sphere of radius R. We are interested in the sequence $\mathbf{r}(k)$ which gives rise to a piecewise linear curve in $S^N(R)$. Our goal is to construct an analog of the Frenet basis along the discrete curve and of the corresponding Frenet equations.

By $F^1(k)$ we denote the 1-dimensional oriented vector subspace of $\mathbf{R}^{N+1}$ given by $\mathbf{r}(k)$ and let $F^{l+1}(k) = F^l(k) + F^1(k+l)$. For the point $\mathbf{r}(k)$ in general position (what we assume in the sequel for simplicity) $F^l(k)$ is l-dimensional oriented subspace of $\mathbf{R}^{N+1}$ ($l \leq N+1$). It is also convenient to denote $F^0(k) = \{0\}$. Now we define the orthonormal Frenet basis $\{\mathbf{f}_l(k)\}_{l=0}^N$ in the point $\mathbf{r}(k)$ of the discrete curve: $\mathbf{f}_l(k)$ is the unit vector of $F^{l+1}(k)$ orthogonal to $F^l(k)$ and correctly oriented.

The distance $\Delta(k)$ between points $\mathbf{r}(k)$ and $\mathbf{r}(k+1)$ of the curve is given in terms of the radius R of the sphere and the angle $\varphi_0(k)$ between $\mathbf{f}_1(k)$ and $\mathbf{f}_1(k+1)$ as

$$\Delta(k) = R\,\varphi_0(k) \quad . \tag{1}$$

We are going to construct $N-1$ other angles which play similar role as curvatures of the (continuous) curve.

Both $F^2(k)$ and $F^2(k+1)$ are subspaces of $F^3(k)$ and their intersection is $F^1(k+1)$. Its orthogonal complement in $F^3(k)$ is the plane $\pi_1(k)$. Since $\mathbf{f}_2(k) \in F^3(k)$ and $\mathbf{f}_2(k) \perp F^2(k)$, then $\mathbf{f}_2(k) \in \pi_1(k)$; similarly $\mathbf{f}_1(k+1) \in \pi_1(k)$. By $\tilde{\mathbf{f}}_1(k)$ we denote the unit vector of $\pi_1(k)$ normal to $\mathbf{f}_2(k)$; it is also the vector of $\pi_0(k) = F^2(k)$ orthogonal to $\mathbf{f}_0(k+1)$. The angle $\varphi_1(k)$ between the hyperplanes $F^2(k)$ and $F^2(k+1)$ in $F^3(k)$ (equivalently, between $\tilde{\mathbf{f}}_1(k)$ and $\mathbf{f}_1(k+1)$) is the angle of geodesic curvature.

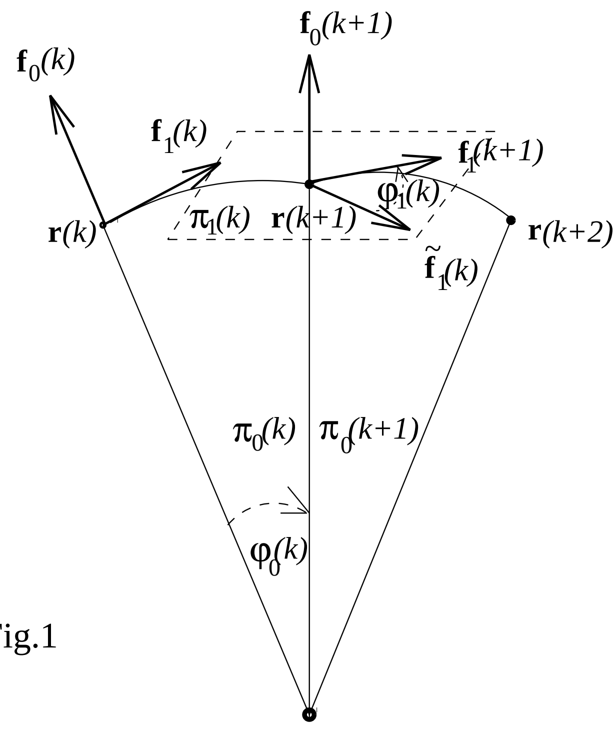

Fig.1

In general we consider the subspaces $F^l(k)$ and $F^l(k+1)$ of $F^{l+1}(k)$, their intersection $F^{l-1}(k+1)$ and its orthogonal complement $\pi_{l-1}(k)$. Since $\mathbf{f}_l(k) \in F^{l+1}(k)$ and $\mathbf{f}_l(k) \perp F^l(k)$, then $\mathbf{f}_l(k) \in \pi_{l-1}(k)$. Similarly, since $\mathbf{f}_{l-1}(k+1) \in F^l(k+1) \subset F^{l+1}(k)$ and $\mathbf{f}_{l-1}(k+1) \perp F^{l-1}(k+1)$, then $\mathbf{f}_{l-1}(k+1) \in \pi_{l-1}(k)$.

By $\tilde{\mathbf{f}}_{l-1}(k)$ we denote the unit vector of $\pi_{l-1}(k)$ orthogonal to $\mathbf{f}_l(k)$. One can show that $\tilde{\mathbf{f}}_{l-1}(k)$ is also the unit vector of $\pi_{l-2}(k)$ orthogonal to $\mathbf{f}_{l-2}(k+1)$. This is the consequence of two facts: $\tilde{\mathbf{f}}_{l-1}(k) \in F^l(k)$ (as $\tilde{\mathbf{f}}_{l-1}(k) \in F^{l+1}(k)$ and $\tilde{\mathbf{f}}_{l-1}(k) \perp \mathbf{f}_l(k)$) and $\tilde{\mathbf{f}}_{l-1}(k) \perp F^{l-1}(k+1)$. The angle $\varphi_{l-1}(k)$ between the hyperplanes $F^l(k)$ and $F^l(k+1)$ of $F^{l+1}(k)$ (equivalently, between their normals $\mathbf{f}_l(k)$ and $\tilde{\mathbf{f}}_l(k)$, or between $\tilde{\mathbf{f}}_{l-1}(k)$ and $\mathbf{f}_{l-1}(k+1)$) is the angle of the $(l-1)$th curvature.

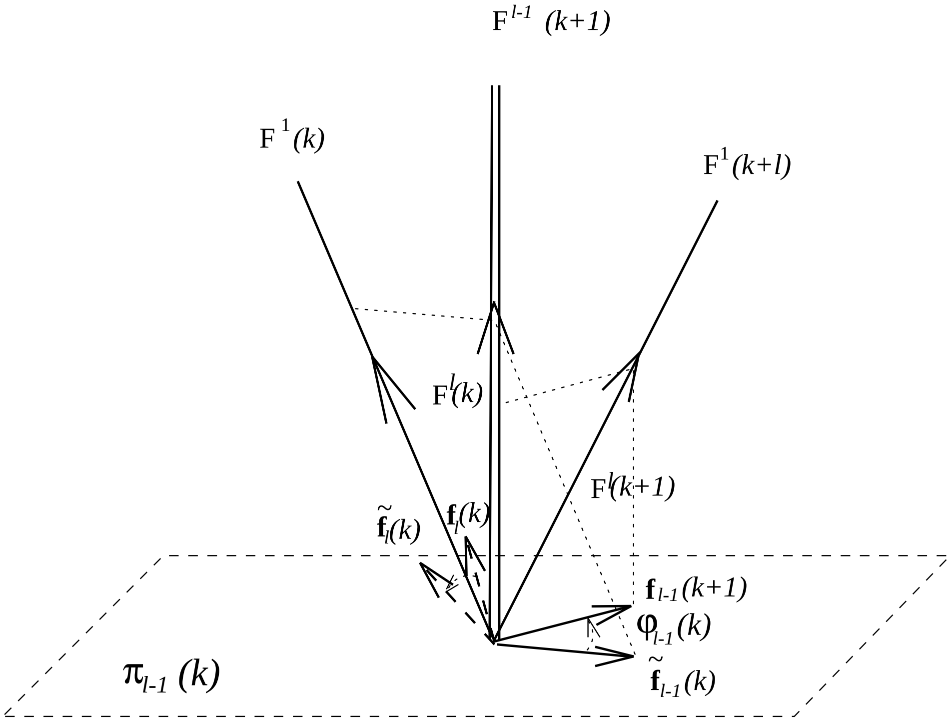

Fig.2

The transition from the Frenet basis $\{\mathbf{f}_l(k)\}_{l=0}^N$ in the point $\mathbf{r}(k)$ of the discrete curve to $\{\mathbf{f}_l(k+1)\}_{l=0}^N$ is obtained by the superposition of N rotations of the angles $\varphi_l(k)$ in the planes $\pi_l(k)$, $(l = 0, ..., N-1)$:

$$\begin{gathered}
\{\mathbf{f}_0(k), \mathbf{f}_1(k), \mathbf{f}_2(k), ..., \mathbf{f}_N(k)\} \stackrel{\varphi_0(k), \pi_0(k)}{\longrightarrow} \\
\{\mathbf{f}_0(k+1), \tilde{\mathbf{f}}_1(k), \mathbf{f}_2(k), ..., \mathbf{f}_N(k)\} \stackrel{\varphi_1(k), \pi_1(k)}{\longrightarrow} ... \\
... \ \{\mathbf{f}_0(k+1), ..., \tilde{\mathbf{f}}_{l-1}(k), \mathbf{f}_l(k), \mathbf{f}_{l+1}(k), ..., \mathbf{f}_N(k)\} \stackrel{\varphi_{l-1}(k), \pi_{l-1}(k)}{\longrightarrow} \\
\{\mathbf{f}_0(k+1), ..., \mathbf{f}_{l-1}(k+1), \tilde{\mathbf{f}}_l(k), \mathbf{f}_{l+1}(k), ..., \mathbf{f}_N(k)\} \stackrel{\varphi_l(k), \pi_l(k)}{\longrightarrow} ... \\
... \ \{\mathbf{f}_0(k+1), ..., \mathbf{f}_{N-1}(k+1), \tilde{\mathbf{f}}_N(k) = \mathbf{f}_N(k+1)\} \ .
\end{gathered} \tag{2}$$

2.2 The spinor representation of the Frenet equations

The linear transformation related to the resulting rotation gives the discrete analog of the Frenet equations. We will present it using the language of the Clifford algebras and Spin groups [27]. Let $\mathcal{E} = \{\mathbf{e}_l\}_{l=0}^N$ be a fixed orthonormal basis of $\mathbf{R}^{N+1}$ cosidered as a subspace of the Clifford algebra $\mathrm{Cl}(N+1)$, then any other orthonormal (correctly oriented) basis

$\mathcal{F} = \{\mathbf{f}_l\}_{l=0}^N$ can be obtained from $\mathcal{E}$ using an element S of the corresponding Spin$(N+1)$ group

$$\mathcal{F} = S^{-1}\mathcal{E}S \quad . \tag{3}$$

Moreover, when another basis $\tilde{\mathcal{F}}$ is obtained from $\mathcal{F}$ by rotation in the plane $\langle \mathbf{f}_i, \mathbf{f}_j \rangle$ of the angle φ, then

$$\tilde{S} = (\cos\frac{\varphi}{2} + \mathbf{e}_i\mathbf{e}_j \sin\frac{\varphi}{2})S = O_{ij}^{\varphi} S \quad . \tag{4}$$

If $S(k) \in$ Spin$(N+1)$ represents the rotation to Frenet basis in point $\mathbf{r}(k)$ then it is subjected to the equation

$$S(k+1) = O_{N-1,N}^{\varphi_{N-1}(k)} \cdot \ldots \cdot O_{12}^{\varphi_1(k)} O_{01}^{\varphi_0(k)} S(k) \quad . \tag{5}$$

The arc-length along the curve

$$s = \sum_i^{k-1} \Delta(i) \tag{6}$$

in the continuous limit $\varphi_0(i) \to 0$ is the arc-length parameter. Moreover

$$\frac{dS(s)}{ds} = \lim_{\varphi_0(k)\to 0} \frac{S(k+1) - S(k)}{\Delta(k)} = \left(\frac{1}{R}E_{01} + \kappa_1(s)E_{12} + \ldots + \kappa_{N-1}(s)E_{N-1,N}\right) S(s) \quad , \tag{7}$$

where

$$\kappa_l(s) = \lim_{\varphi_0(k)\to 0} \frac{\varphi_l(k)}{\Delta(k)} \tag{8}$$

is the l-th curvature of the corresponding continuous curve and $E_{ij} = \mathbf{e}_i\mathbf{e}_j/2$ are elements of the canonical basis of the orthogonal Lie algebra so$(N+1)$ in the Clifford algebra representation. The equation (7) is nothing but the classical Frenet equation for the curve in $S^N(R)$.

Remark: Even when point $\mathbf{r}(k)$ is not in general position, the corresponding spaces $F^l(k)$ can be defined in a way that their dimension is l. We just keep the space $F^l(k-p)$ $(p > 0)$ of the nearest point in which it was "properly" defined. In this way one can define, for example, the Frenet frame along the geodesic line (which is any big circle): only $\mathbf{f}_0$ and $\mathbf{f}_1$ vary along the curve, and the rest of the Frenet frame remains as it was defined in a starting point.

3 The discrete curve in $S^3(R)$

In this Section we investigate in detail the discrete curve in $S^3(R)$ with constant distance Δ between the subsequent points. Throught the Section we use the following definitions: $\nu = \Delta/R = \varphi_0(k)$, $\varphi(k) = \varphi_1(k)$, $\theta(k) = \varphi_2(k)$, the Frenet basis $\{\mathbf{f}_l(k)\}_{l=0}^3$ is denoted by $\{\hat{\mathbf{r}}(k), \mathbf{t}(k), \mathbf{n}(k), \mathbf{b}(k)\}$ and consists of the radial, tangent, normal and binormal vectors.

We show that the Frenet equation (5) reduces to the Ablowitz-Ladik spectral problem, and its continuous limit to the Zakharov-Shabat spectral problem. We also present the geometric explanation of Sym's formula.

3.1 The Hasimoto transformation for the discrete curve in S^3 and the Ablowitz-Ladik spectral problem

It is convenient to modify Frenet basis by a rotation in the normal plane $\langle \mathbf{n}(k), \mathbf{b}(k)\rangle$ of the angle $\sigma(k) = \sum_i^{k-1} \theta(i)$.

$$\mathbf{N}(k) = \cos\sigma(k)\mathbf{n}(k) - \sin\sigma(k)\mathbf{b}(k) \tag{9}$$

$$\mathbf{N}_J(k) = \sin\sigma(k)\mathbf{n}(k) + \cos\sigma(k)\mathbf{b}(k) \ \ .$$

This change of basis corresponds to a partial "integration" of the Frenet equations in the normal plane, since the vectors $\mathbf{N}(k), \mathbf{N}_J(k)$ does not vary from the point of view of the normal plane (this is the discrete analog of the parallel transport in the normal bundle).

It is also convenient to interprete any vector of the normal plane as a complex number

$$\vec{\phi}(k) = \mathrm{Re}\phi(k)\mathbf{N}(k) + \mathrm{Im}\phi(k)\mathbf{N}_J(k) \quad \Leftrightarrow \quad \phi(k) = \mathrm{Re}\phi(k) + i\ \mathrm{Im}\phi(k) \ \ . \tag{10}$$

If we define $S(k) \in \mathrm{Spin}(4)$ by the relation

$$\mathcal{H}(k) = \{\hat{\mathbf{r}}(k), \mathbf{t}(k), \mathbf{N}(k), \mathbf{N}_J(k)\} = S(k)^{-1}\mathcal{E}S(k) \ \ ,$$

then

$$S(k+1) = O_{23}^{-\sigma(k)} O_{12}^{\varphi(k)} O_{23}^{\sigma(k)} O_{01}^{\nu} S(k) \ \ .$$

To represent the above rotation in terms of matrices we first choose the following representation of the basis $\mathcal{E}$ as 4×4 Dirac matrices

$$\mathbf{e}_0 \leftrightarrow \begin{pmatrix} 0 & \mathrm{I} \\ \mathrm{I} & 0 \end{pmatrix} \ \ , \ \ \mathbf{e}_1 \leftrightarrow \begin{pmatrix} 0 & -i\sigma_3 \\ i\sigma_3 & 0 \end{pmatrix} \ \ , \tag{11}$$

$$\mathbf{e}_2 \leftrightarrow \begin{pmatrix} 0 & -i\sigma_1 \\ i\sigma_1 & 0 \end{pmatrix} \ \ , \ \ \mathbf{e}_3 \leftrightarrow \begin{pmatrix} 0 & i\sigma_2 \\ -i\sigma_2 & 0 \end{pmatrix} \ \ ,$$

where I is the 2×2 identity matrix and σ_l are the standard Pauli matrices . The linear problem related to the discrete curve takes the form

$$S(k+1) = \begin{pmatrix} S'(k+1) & 0 \\ 0 & S''(k+1) \end{pmatrix} = \begin{pmatrix} A'(k) & 0 \\ 0 & A''(k) \end{pmatrix} \begin{pmatrix} S'(k) & 0 \\ 0 & S''(k) \end{pmatrix} \ \ , \tag{12}$$

with

$$A'(k) = \begin{pmatrix} \mathrm{e}^{i\nu/2}\cos(\varphi(k)/2) & \mathrm{e}^{-i\nu/2}\sin(\varphi(k)/2)\mathrm{e}^{i\sigma(k)} \\ -\mathrm{e}^{i\nu/2}\sin(\varphi(k)/2)\mathrm{e}^{-i\sigma(k)} & \mathrm{e}^{-i\nu/2}\cos(\varphi(k)/2) \end{pmatrix} =$$

$$= \frac{1}{\sqrt{1+|q(k)|^2}} \begin{pmatrix} \zeta & q(k)\zeta^{-1} \\ -\bar{q}(k)\zeta & \zeta^{-1} \end{pmatrix} \ \ , \tag{13}$$

wher

$$q(k) = \tan(\varphi(k)/2)\mathrm{e}^{i\sigma(k)} \ \ , \ \ \zeta = \mathrm{e}^{i\nu/2} \ \ , \tag{14}$$

and $A''(k,\zeta) = A'(k,\zeta^{-1})$. This linear problem is equivalent [22] to the Ablowitz-Ladik spectral problem [23].

In the limit of the continuous curve we obtain the Zakharov-Shabat spectral problem [10]

$$\frac{dS'(s)}{ds} = \frac{1}{2} \begin{pmatrix} i\lambda & q(s) \\ -\bar{q}(s) & -i\lambda \end{pmatrix} S'(s) \ \ , \tag{15}$$

where $q(s) = \kappa(s)\mathrm{e}^{i\sigma(s)}$, $\sigma(s) = \int^s \tau(s')ds'$ and $\lambda = R^{-1}$.

3.2 The geometric interpretation of Sym's formula

In short notation: $\mathcal{E} \leftrightarrow \{\mathrm{I}, i\sigma_3, i\sigma_1, -i\sigma_2\}$

$$\mathcal{H}(k) = S''(k)^{-1}\mathcal{E}S'(k) \tag{16}$$

and, as a consequence, for a function $q(k)$, the radius vector of the corresponding curve in sphere $S^3(R)$ is represented by

$$\mathbf{r}(k) = R\,S''(k)^{-1}\,S'(k) \ . \tag{17}$$

Suppose one is interested in the radius vector of the curve in $\mathbf{R}^3$ corresponding to $q(k)$. One can consider $\mathbf{R}^3$ as sphere of infinit radius but one cannot just take the limit $R \to \infty$ in the formula above. This way the center of the sphere is fixed while $\mathbf{R}^3$ is pushed away to infinity.

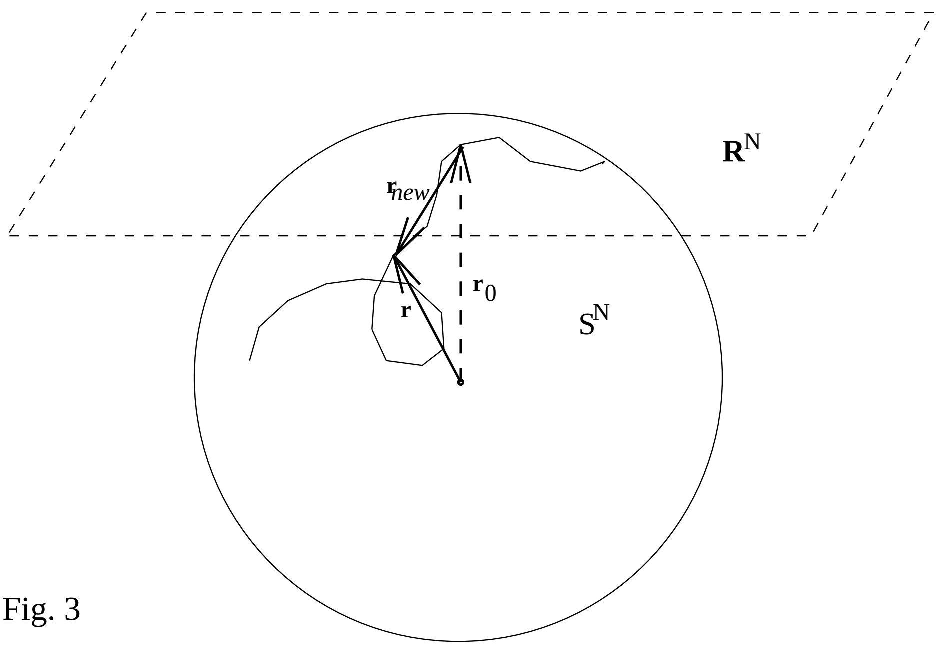

Fig. 3

To remove this incovenience we first have to shift the basis of $\mathbf{R}^4$ to a point of the sphere: $\mathbf{r}_{new}(k) = \mathbf{r}(k) - \mathbf{r}_0$.

If the linear problem is solved under the initial condition $S'(0,\lambda) = S''(0,\lambda) = \mathrm{I}$, then $S''(k,\lambda) = S'(k,-\lambda)$. Choosing $\mathbf{r}_0 = \mathbf{r}(0,\lambda) = R\mathbf{e}_0 = \frac{1}{\lambda}\mathrm{I}$, one obtains the following formula for the Cartesian coordinates $(X^i(k))_{i=1}^3$ of the points $\tilde{\mathbf{r}}(k)$ of the curve in $\mathbf{R}^3$

$$\tilde{\mathbf{r}}(k) = \sum_{i=1}^{3} X^i(k)\mathbf{e}_i = \lim_{\lambda\to 0}\frac{1}{\lambda}\left(S'(k,\lambda)^{-1}S'(k,\lambda) - \mathrm{I}\right) = 2S'(k,0)^{-1}\frac{\partial S'(k,\lambda)}{\partial\lambda}|_{\lambda=0} \ . \tag{18}$$

The above formula was first used by Sym [12] in his approach of soliton surfaces. He also found its generalization to soliton equations related to linear problems in semi-simple Lie algebras. Our approach is more related to the Clifford algebras.

Remark: A modification of this formula was recently found by Cieśliński [28] in the context of conformal geometry of isothermic surfaces in $\mathbf{R}^3$.

4 The integrable dynamics in $S^3(R)$

In this Section we consider the motion of the discrete curve subjected to *Properties 1 - 3*. For convenience, we use the following short-hand notation: f for $f(k), k \in \mathbf{Z}$ and f_n for $f(k+n),\ n = \pm 1, \pm 2, ...$

The motion of the curve is governed by velocity field $\mathbf{v}$ which is convenient to write in the form

$$\mathbf{r}_{,t} = \mathbf{v} = \frac{\sin(\lambda\Delta)}{\lambda} \left(V\mathbf{t} + \mathrm{Re}\phi\mathbf{N} + \mathrm{Im}\phi\mathbf{N}_J \right) \quad , \tag{19}$$

and, consequently,

$$S'_{,t} = TS' = \frac{i}{2} \begin{pmatrix} \gamma & \delta \\ \bar{\delta} & -\gamma \end{pmatrix} S' \quad , \quad T \in \mathrm{su}(2) \quad . \tag{20}$$

The compatibility condition between equations (12)(19) and (20) specifies the entries γ and δ in terms of the velocity field:

$$\begin{aligned} \delta &= i\zeta^{-2}\phi - i(\phi_1 + q(V + V_1)) \\ \gamma &= W + \sin(\lambda\Delta)V \\ W_1 - W &= -\mathrm{Re}\left(i\bar{q}\left(\phi_2 - \phi + q_1(V_1 + V_2) \right) \right) \end{aligned} \tag{21}$$

and yields the kinematics

$$2q_{,t} = -2\cos(\lambda\Delta)\left(\phi_1 + qV_1\right) + \mathcal{R}\left(\phi_1 + qV_1\right) \ , \tag{22}$$

$$(1 - |q^2|)V_1 - (1 + |q|^2)V = q\bar{\phi}_1 + \bar{q}\phi_1 \quad , \tag{23}$$

where

$$\mathcal{R}f := (1+|q|^2)\left(f_1 + f_{-1} + 2(q_1E - q_{-1})(E-1)^{-1}\mathrm{Re}\frac{\bar{q}f}{1+|q|^2} \right) - 2iq(E-1)^{-1}\mathrm{Im}(\bar{q}_1 f - \bar{q}f_1) \tag{24}$$

and E is the shift operator along the discrete curve: $Ef = f_1$.

Substituting the ansatz

$$\begin{pmatrix} V \\ \phi \end{pmatrix} = \sum_{j=0}^{m} (\cos(\lambda\Delta))^j \begin{pmatrix} V^{(m-j)} \\ \phi^{(m-j)} \end{pmatrix} \tag{25}$$

into equation (22) and requiring independence of $\cos(\lambda\Delta)$, we finally obtain the following class of integrable dynamics:

$$q_{,t} = h_0(\mathcal{R})(1 + |q|^2)(q_1 - q_{-1}) + h_1(\mathcal{R})(iq) \quad , \tag{26}$$

where h_0 and h_1 are arbitrary entire functions with real coefficients.

We remark that equation (22) implies the following interesting connection:

$$K^{(m)} = \phi_1^{(m)} + qV_1^{(m)} \tag{27}$$

between the integrable commuting flows

$$K^{(m)} = \mathcal{R}^{m-1}(1 + |q|^2)(q_1 - q_{-1}) \ \text{ and/or } \ K^{(m)} = \mathcal{R}^m(iq) \quad , \quad m \geq 0 \tag{28}$$

and the velocity fields. In the continuous limit this reduces to the result of Langer and Perline [14].

The simplest examples are the following:

i) If $h_0 = 1$, $h_1 = 0$, then

$$\mathbf{v} = \frac{\sin(\lambda\Delta)}{\lambda}(\mathbf{t} - \vec{q}_{-1}) = \frac{\sin(\lambda\Delta)}{\lambda}(\mathbf{t} - |q_{-1}|\mathbf{n}) \quad , \tag{29}$$

$$q_{,t} = (1 + |q|^2)(q_1 - q_{-1}) =: K^{(1)} \quad . \tag{30}$$

ii) If $h_0(x) = x$, $h_1 = 0$, then

$$\mathbf{v} = \frac{4\sin(\lambda\Delta)}{\lambda}\left(\left(\cos(\lambda\Delta) + \frac{1}{2}(q\bar{q}_{-1} + \bar{q}q_{-1})\right)\mathbf{t} + \vec{\phi}\right) \quad , \tag{31}$$

$$\phi = \left(-q_{-1}\cos(\lambda\Delta) + \frac{1}{2}\left((1 + |q_{-1}|^2)(q - q_{-2}) - q_{-1}(q\bar{q}_{-1} + \bar{q}q_{-1})\right)\right)$$

$$q_{,t} = (1 + |q|^2)\left((1 + |q_1|^2)q_2 - (1 + |q_1|^2)q_{-2} + \bar{q}(q_1^2 - q_{-1}^2) + q(q_1\bar{q}_{-1} - q_{-1}\bar{q}_1)\right) =: K^{(2)} \quad . \tag{32}$$

iii) If $h_0 = 0$ and $h_1(x) = x$, we obtain

$$\mathbf{v} = \frac{2\sin(\lambda\Delta)}{\lambda}(i\vec{q}_{-1}) = \frac{2\sin(\lambda\Delta)}{\lambda}\tan(\frac{\varphi_{-1}}{2})\mathbf{b} \quad , \tag{33}$$

$$q_{,t} = i(1 + |q|^2)(q_1 + q_{-1}) \quad . \tag{34}$$

This equation has also recently appeared in conection with the Heisenberg XXO antiferromagnet model [29].

In the continuous limit, $\mathcal{R} - 2$ reduces to the recursion operator of the continuous NLS hierarchy. Moreover the following combination of equation (34) with the "zero order flow" $q_{,t} = iq$:

$$q_{,t} = i\left(q_1 - 2q + q_{-1} + |q|^2(q_1 + q_{-1})\right) \tag{35}$$

reduces [23] in the continuous limit, to the NLS equation

$$iq_{,t'} = q_{,ss} + \frac{1}{2}|q|^2 q \quad , \quad t' = -\Delta^2 t \quad . \tag{36}$$

which describes the motion of a vortex filament in the localized induction approximation [5][30]. We remark that, in this approximation, the velocity field which governs the motion of the vortex depends on its curvature κ through the relation

$$\mathbf{r}_{,t} = \kappa\mathbf{b} \quad ; \tag{37}$$

therefore, since equation (35) has in $\mathbf{R}^3$ a velocity field of the same type:

$$\mathbf{r}_{,t} = 2\Delta\tan(\frac{\varphi_{-1}}{2})\mathbf{b} \quad , \tag{38}$$

we expect it to be a good candidate for describing the motion of a discrete vortex in the same approximation.

Consequently, the continuous limit of the linear combination

$$q_{,t} = K^{(2)} - 2K^{(1)} \tag{39}$$

reduces to the complex mKdV equation

$$q_{,t'} = q_{,sss} + \frac{3}{2}|q|^2 q_{,s} \ \ , \ \ t' = 2\Delta^3 t \ \ . \tag{40}$$

We remark that, in the degenerate case of the curve on S^2, $\theta \equiv 0$ and consequently $q, \phi \in \mathbf{R}$. In this case we are forced to choose $h_1 = 0$ and only the first hierarchy survives.

Property 3 can also be satisfied through a mechanism (different from that of equation (25)) which gives integrable dynamics with sources (see [21][22]).

5 Discrete-time dynamics and the Hirota equation

In this section we consider the discrete curve on $S^2(R)$ with constant distance $\Delta = \nu R$ between subsequent points "moving" in discrete time.

Now we use the following notation: f for $f(k,l)$, $k, l \in \mathbf{Z}$, and f_m^n for $f(k+m, l+n)$. The Frenet equations read

$$S_1 = O_{12}^{\varphi} O_{01}^{\nu} S = AS \ \ . \tag{41}$$

The discrete-time kinematics can be described in terms of angles ω and μ which are analogues of the direction and the length of the velocity field respectively.

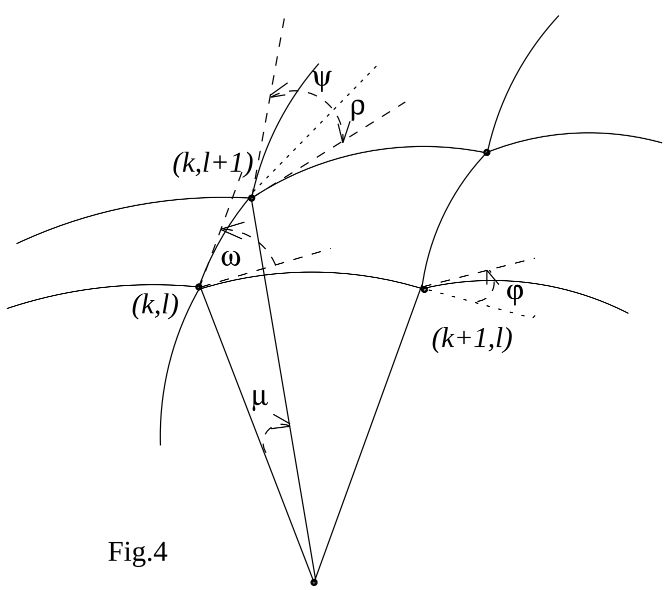

Fig.4

The induced kinematics of the Frenet frame reads

$$S^1 = O_{12}^{\rho} O_{01}^{\mu} O_{12}^{\omega} S = BS \ \ , \tag{42}$$

where the angle ρ should be calculated from the compatibility condition

$$A^1 B = B_1 A \ \ . \tag{43}$$

It turns out that it is convenient to use, instead of ρ, another angle $\psi = \omega^1 + \rho$, which is the angle of curvature of the curve $\{\mathbf{r}^n\}_{n\in\mathbf{Z}}$.

The compatibility condition (43) written in terms of angles splits into three equations

$$\begin{gathered}
\cos\frac{\mu_1}{2}\cos\frac{\omega_1+\varphi-(\omega_1^1+\varphi^1-\psi_1)}{2}=\cos\frac{\mu}{2}\cos\frac{\omega-(\omega^1-\psi)}{2}\quad ,\\
\sin\frac{\mu_1}{2}\cos\frac{\omega_1+\varphi+\omega_1^1+\varphi^1-\psi_1}{2}=\cos\frac{\mu}{2}\cos\frac{\omega+\omega^1-\psi}{2}\quad ,\\
\sin\frac{\mu_1}{2}\sin\frac{\omega_1+\varphi+\omega_1^1+\varphi^1-\psi_1}{2}+i\cos\frac{\mu_1}{2}\sin\frac{\omega_1+\varphi-(\omega_1^1+\varphi^1-\psi_1)}{2}=\\
\mathrm{e}^{-i\nu}\left(\sin\frac{\mu}{2}\sin\frac{\omega+\omega^1-\psi}{2}+i\cos\frac{\mu}{2}\sin\frac{\omega-(\omega^1-\psi)}{2}\right)\quad .
\end{gathered}\tag{44}$$

In this paper we consider only the motion subjected to the condition $\mu\equiv$const. This is (together with the previous condition $\nu\equiv$const.) the discrete analog of the Tchebyschev net condition, which in the continuous case gives the sine-Gordon equation [31]. The first two equations of (44) imply

$$\varphi_{-1}^1-\varphi_{-1}^{-1}=-(\psi_1^{-1}-\psi_{-1}^{-1})\tag{45}$$

which asserts the existence of the "potential" ϕ:

$$\varphi=\phi_2-\phi\quad ,\quad \psi=-\phi^2+\phi\quad ,\tag{46}$$

and allows to write ω in terms of it

$$\omega=-(\phi^1+\phi_1)\quad .\tag{47}$$

Finally, the third equation of (44) gives the celebrated Hirota equation [24]

$$\sin\frac{\phi^1+\phi_1-\phi_1^1-\phi}{2}=\tan\frac{\mu}{2}\tan\frac{\nu}{2}\sin\frac{\phi^1+\phi_1+\phi_1^1+\phi}{2}\quad .\tag{48}$$

References

[1] J. Liouville, J. Math. Pures Appl. 18 (1853) 71.

[2] L. Bianchi, *Applicazioni goemetriche del metodo delle approssimazioni successive di Picard*, in: *Opere*, Vol. I, Part 2, p. 137. Edizioni Cremonese, Roma 1953.

[3] L. Bianchi, *Lezioni di geometria differenziale*, Vol.1, Spoerri, Pisa, 1892.

[4] A. V. Bäcklund, *Concerning Surfaces with Constant Negative Curvature*, New Era Printing Co., Lancaster, Pa., 1905.

[5] H. Hasimoto, J. Fluid Mech. 51 (1972) 477.

[6] G. L. Lamb, J. Math. Phys. 18 (1977) 1654.

[7] M. Lakshmanan, Th. W. Ruijgrok and C. J. Thompson, Physica 84 A (1976) 577.
M. Lakshmanan, Phys. Lett. 64 A (1978) 353.
M. Lakshmanan, J. Math. Phys. 20 (1979) 1667.

[8] R. Sasaki, Nuc. Phys. B 154 (1979) 343

[9] S. S. Chern and K. Tenenblat, J. Diff. Geom. 16 (1981) 347.

[10] V.E. Zakharov and A.B. Shabat, JETPh 61 (1971) 118.

[11] M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur, Stud. Appl. Math. 53 (1974) 249.

[12] A. Sym, *Soliton theory is surface theory*, IFT UW 11/81, Warsaw University.
A. Sym, *Soliton Surfaces and Their Applications*, in: Lecture Notes in Physics, Vol. 239 p.154, Springer, Berlin 1985.

[13] A. I. Bobenko, Math. Ann. 290 (1990) 207.

[14] J. Langer and R. Perline, J. Nonlinear Sci. 1 (1991) 71.

[15] R. E. Goldstein and D. M. Petrich, Phys. Rev. Lett. 67 (1991) 3203.

[16] K. Tenenblat and C. L. Terng, Ann. of Math. 111 (1980) 477.
C. L. Terng, Ann. of Math. 111 (1980) 491.

[17] B.G. Konopelchenko, *Induced surfaces and their integrable dynamics*, Budker INP 93-114.

[18] W. Wunderlich, Ak. Wiss. Wien 160 (1951) 39.

[19] A. Bobenko and U. Pinkall, *Discrete Surfaces with Constant Negative Gaussian Curvature and the Hirota Equation*, SFB 288 Preprint No. 127, 1994.

[20] U. Pinkall, *Discrete Harmonic Maps and Integrable Systems*, lecture given at the conference *Harmonic Maps and Integrable Systems*, Leeds 1992.

[21] A. Doliwa and P. M. Santini, Phys. Lett. A 185 (1994) 373.

[22] A. Doliwa and P. M. Santini, *Integrable dynamics of a discrete curve and the Ablowitz-Ladik hierarchy*, IFT UW 6/94, Warsaw University, to appear in Journal of Mathematical Physics.

[23] M. J. Ablowitz and J. F. Ladik, J. Math. Phys. 16 (1975) 598.
M. J. Ablowitz and J. F. Ladik, J. Math. Phys. 17 (1976) 1011.

[24] R. Hirota, J. Phys. Soc. Jap. 43:6 (1977) 2079.

[25] S. J. Orfanidis, Phys. Rev. 18 D, (1978) 3828.

[26] D. Levi and R. Benguria, Proc. Nat. Acad. Sci. (USA) 77 (1980) 5025.

[27] P. Budinich and A. Trautman, *The Spinorial Chessboard*, Springer, Berlin, 1988.

[28] J. Cieśliński, *The Darboux-Bianchi transformation for isothermic surfaces. Classical results versus the soliton approach*, IP-WUD 15/94, Białystok.

[29] A. R. Its, A. G. Izergin, V. E. Korepin and N. A. Slavnov, *Temperature Correlations of Quantum Spins*, INS 211, Clarkson University, 1992.

[30] G. K. Batchelor, *An Introduction to Fluid Mechanics*, Cambridge University Press, New York, 1967.

[31] Tian Chou, Chin. Ann. Math. B 9 (1988) 118.